Application of Time-Iterative Schemes to Incompressible Flow

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The computation of steady incompressible flows by an Euler implicit algorithm is studied using both the incompressible equations and the low Mach number compressible equations. The incompressible equations are handled by adding an artificial time derivative to the continuity equation. This allows both the pressure and velocity to be obtained implicitly. In one-dimensional problems, both systems converge rapidly, even at low Mach numbers where the eigenvalues are very stiff. In two dimensions where approximate factorization is required, the presence of stiff eigenvalues is highly detrimental. Stiffness can be avoided in the incompressible equations by selecting an appropriate "pseudo"-Mach number. This insures reliable convergence and results in an efficient incompressible flow algorithm. In the case of compressible equations, the Mach number cannot be chosen arbitrarily and the contamination introduced by approximate factorization must be removed. A matrix preconditioning factor that accomplishes this is developed and demonstrated. With this modification, the convergence rate is the same as in the incompressible case and is independent of Mach number. Rapid convergence is observed at Mach numbers as low as 0.05.

Introduction

THE development of computational algorithms for compressible flows has advanced at a rapid pace in the past several years. The advent of reliable time-dependent algorithms representing joint applications of basic mathematical concepts and fundamental physical principles has been largely responsible for this improvement.

The computation of incompressible flows is not as well understood. Although some progress has been realized, it has not been as rapid as for compressible flows. Perhaps the most characteristic aspect of the computation of incompressible flows is the difficulty of extracting the pressure from the combined continuity and momentum equations. One common procedure is to define a Poisson equation[1,2] or a specially formulated "correction" equation[3,4] for the pressure. The coupling between these special equations and the remaining conservation law frequently has an adverse effect on convergence.

It is the purpose of this paper to investigate the application of implicit algorithms that have been developed for compressible flows to the computation of steady, incompressible flowfields. Incompressible solutions will be attempted in two ways: by applying compressible algorithms to the incompressible equations and by using these algorithms on the compressible equations in the low Mach number limit. The compressible equations are required for flowfields where compressibility effects are present in some local regions, but are absent in others. The compressible equations could also be used for completely incompressible flowfields if it could be shown that they lead to more reliable, robust, or economical algorithms. Such, however, is not the case. The use of implicit algorithms with the incompressible equations is shown to yield the same convergence rates as are observed with the compressible equations, but to require substantially less CPU time per step. One advantage of studying incompressible flows with both compressible and incompressible equations is that each formulation can be used as an aid in understanding and improving the convergence characteristics of the other.

It is well known that the convergence of implicit schemes is woefully slow (or lacking altogether) in compressible flows with low Mach numbers. In the present paper, a technique for alleviating this problem is identified by observing the convergence properties of the incompressible equations when similar "compressible" algorithms are used.

Application of compressible algorithms to be incompressible equations is accomplished by adding a time derivative to the continuity equation in a manner analogous to that originally suggested by Chorin[5] and later implemented in an implicit algorithm by Steger and Kutler.[6] More recent applications have also been reported by Chang and Kwak.[7,8] The addition of a time derivative to the continuity equation transforms the incompressible equations to a totally hyperbolic system that can be solved by standard, implicit, time-marching methods.[9,10] This approach circumvents the traditional problems in the incompressible calculations noted above.

Although our interests lie in high Reynolds number flows, the present paper is limited to the inviscid equations. The presence of small diffusive terms has little effect on convergence when the convective terms are handled in a proper, stable manner, as evidenced by viscous calculations[8] and stability analyses[11] reported elsewhere.

Problem Formulation

The Compressible and Incompressible Equations

The unsteady form of the compressible Euler equations is hyperbolic in time. The incompressible equations are also rendered hyperbolic by adding an artificial time derivative to the continuity equation. With such an artificial compressibility term added, we can express both the compressible and the incompressible formulations in generalized coordinates, ξ and η, by the vector system

$$\frac{\partial Q}{\partial t}+\frac{\partial E}{\partial \xi}+\frac{\partial F}{\partial \eta}=0 \tag{1}$$

which, for simplicity, we have restricted to two dimensions. When Eq. (1) is applied to compressible flows, we designate the vectors Q, E, and F by the subscript c, which take on

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their standard form,

$$Q_c = J^{-1}(\rho,\rho u,\rho v,e)^T$$

$$E_c = J^{-1}(\rho U,\rho uU + \xi_x p,\rho vU + \xi_y p,(e+p)U)^T$$

$$F_c = J^{-1}(\rho V,\rho uV + \eta_x p,\rho vV + \eta_y p(e+p)V)^T \tag{2}$$

For the incompressible case, we designate these vectors by the subscript i, and they become

$$Q_i = J^{-1}(p/\beta,u,v)^T$$

$$E_i = J^{-1}(U,uU + \xi_x p,vU + \xi_y p)^T$$

$$F_i = J^{-1}(V,uV + \eta_x p,vV + \eta_y p)^T \tag{3}$$

In Eqs. (2) and (3), J is the Jacobian of the transformation; u and v are the Cartesian velocity components, while U and V are the contravariant velocities; p and ρ are pressure and density (for incompressible flow the pressure has been reduced by the density); and e is the total energy defined for compressible flows as

$$e = \rho[\epsilon + (u^2 + v^2)/2] \tag{4}$$

where ϵ is the internal energy. The compressible formulation must also be augmented by an equation of state, which is here taken as that of a perfect gas, $p = \rho RT$.

As can be seen from Eq. (3), the unsteady term added to the continuity equation is $(1/\beta)\partial p/\partial t$. The quantity β is chosen to accelerate convergence. Maximum convergence speed (in two dimensions) was found to correspond to a value of $\beta = 1$. Similar conclusions were also reached by Chang and Kwak.[7] If transient solutions were desired, β would have to be chosen large, but for the present steady solutions, the value of β can be selected to enhance convergence.

Discretization and Solution of the Equations of Motion

Starting from the vector form of Eq. (1), an Euler-implicit scheme of the Beam-Warming,[9] Briley-McDonald[10] type can be applied to give

$$\left(I + \Delta t\,\frac{\partial A}{\partial \xi}\right)\left(I + \Delta t\,\frac{\partial B}{\partial \eta}\right)\Delta Q = -\Delta t R \tag{5}$$

where all indicated differentiations are to be replaced by central differences. In Eq. (5), R is the residual of the steady equation at the previous time level and ΔQ represents the change in the dependent variable. The Jacobian matrices A and B in Eq. (5) are given in numerous places for the compressible equations (see, for example, Ref. 9). For incompressible equations, they are given in Ref. 6.

Boundary Conditions

Our experience indicates that the use of implicit boundary procedures is critical to achieving the most rapid convergence over the widest range of conditions. In the implicit procedure chosen,[12,13] Eq. (5) is transformed to characteristic form by premultiplying by a matrix S (which contains the left eigenvectors of A or B). The information from the outgoing characteristics is obtained from this modified version of Eq. (5), while the information on the incoming characteristics is replaced by a Taylor's series expansion of the specified boundary conditions in the form $\Omega(Q) = 0$ to give

$$\frac{\partial \Omega}{\partial Q}\Delta Q = 0 \tag{6}$$

The left eigenvectors of the compressible equations are again given elsewhere.[12,13] The incompressible version is

$$S_i = \begin{Bmatrix} \dfrac{v\xi_x - u\xi_y}{c_i^2} & \dfrac{vU + \beta\xi_y}{c_i^2} & \dfrac{-uU - \beta\xi_x}{c_i^2} \\[2ex] -\dfrac{1}{2c_i^2} & \dfrac{-(U+c_i)\xi_x}{2c_i^2(\xi_x^2+\xi_y^2)} & \dfrac{-(U+c_i)\xi_y}{2c_i^2(\xi_x^2+\xi_y^2)} \\[2ex] \dfrac{1}{2c_i^2} & \dfrac{(U-c_i)\xi_x}{2c_i^2(\xi_x^2+\xi_y^2)} & \dfrac{(U-c_i)\xi_y}{2c_i^2(\xi_x^2+\xi_y^2)} \end{Bmatrix} \tag{7}$$

where

$$c_i = [U^2 + \beta(\xi_x^2 + \xi_y^2)]^{1/2} \tag{8}$$

Here, c_i acts as a sound speed in the ξ direction. As can be seen, c_i is always greater than U (i.e., the pseudo-Mach number U/c_i is always less than 1) if β is positive.

The left eigenvectors of the Jacobian matrix B_i can be obtained from Eq. (7) by replacing ξ_x, ξ_y, and U with η_x, η_y, and V. This substitution also gives the sound speed in the η direction.

For an inflow boundary, two eigenvalues of the matrix A_i are positive and so two boundary conditions must be imposed. For the present calculation, these are chosen as constant total pressure, $p^\circ = p + (u^2 + v^2)/2$, and specified flow angle, $V/U = 0$. For an outflow boundary, only one boundary condition can be imposed. This boundary condition was chosen to be the constant static pressure. When the compressible equations were used, identical boundary conditions were chosen: constant stagnation pressure, zero flow angle, and downstream static pressure. Because of the additional equation in the compressible system, the stagnation enthalpy was also specified at the inlet.

On the wall, only the $V - c_i$ eigenvalue of the matrix B_i propagates into the flowfield, so only one boundary condition can be imposed. Here, we specify the contravariant velocity, $V = 0$. At the centerline, the symmetry condition is used in lieu of a boundary condition.

Computational Results

The formulations described above have both been used to compute the flow through a converging-diverging passage. Results for both one- and two-dimensional calculations are presented. The one-dimensional calculations allow us to study the effects of the stiffness introduced by large values of β or by low Mach numbers apart from the presence of approximate factorization effects. These two phenomena are then encountered simultaneously in the two-dimensional calculations. We begin by presenting the one-dimensional results and then proceed to the two-dimensional solutions.

Convergence of the One-Dimensional Algorithm

Both the compressible and the incompressible versions of the quasi-one-dimensional equations were solved for a converging-diverging passage whose throat area was 90% of its inlet area. Additional geometric details are given later with the two-dimensional solutions. Computations were made for a wide range of values of β and Mach number. For conciseness, only the convergence rates for one pair of calcuations are presented here. Figure 1 shows the convergence of the compressible equations for a nozzle inlet Mach number of 0.001 and that of the incompressible equations based upon a choice of $\beta = 10^4$ (a pseudo-Mach number of 0.01). These results are presented in terms of the L_2 norms of the residuals of each equation as a function of the time step and are for calculations based on 55 grid points. Although both sets of equations are extremely stiff, they each converge to machine accuracy in less than 20 iterations. The final solutions for the compressible and incompressible formulations are identical.

Other calculations with $\beta = 1$ and higher entry Mach numbers (with unchoked throat) indicated that this rapid convergence could be maintained for a wide range of parameters. Clearly, the stiffness of the equations has no effect on one-dimensional calculations and we conclude that the Euler implicit algorithm can be used effectively with either the compressible or the incompressible equations to compute incompressible, one-dimensional flow.

Two-Dimensional Calculations

In two-dimensional calculations where approximate factorization is needed, the stiff eigenvalues are much more troublesome. Here, the test case chosen was the cascade geometry used by Ni[14] and Chima and Johnson.[15] The geometry and the 65×17 grid pattern (also the same as that used in Refs. 14 and 15) are shown in Fig. 2. Solutions were again computed with both the compressible and incompressible equations using several different inlet Mach numbers and several values of β. The deleterious effect of decreasing Mach number on the convergence of the compressible equations is quickly seen. Figure 3 shows that the convergence of the Euler implicit scheme at an inlet Mach number of 0.5 is reasonably rapid. (The converged results at $M = 0.5$ are in excellent agreement with explicit calculations of Ref. 15.) When the Mach number is lowered to the nearly incompressible value of 0.1, the convergence becomes extremely slow, as can be seen from Fig. 4. The reasons for this deterioration in convergence and a method for circumventing it are described later.

The convergence of the incompressible equations for flow through the same cascade is presented in Fig. 5. By comparison with Fig. 4, we see that with $\beta = 1$ the incompressible scheme converges slightly better than the compressible scheme did for a Mach number of 0.5. The $\beta = 1$ condition, however, corresponds to a pseudo-Mach number of 0.707 at the inlet, which is somewhat larger than that used for the compressible results. Additional calculations with the compressible equations showed that, as the Mach number was increased toward 0.707, their convergence improved to that observed with the incompressible equations. The compressible calculations reported herein were kept at lower Mach numbers to avoid the transonic effects that have no counterpart in the incompressible calculations.

When the value of β in the incompressible equations was increased, the convergence of these equations also became much slower. Calculations with $\beta = 100$ and 10^4 exhibited convergence rates analogous to those shown in Fig. 4 for the low Mach number compressible calculations. Additional calculations for $\beta < 1$ indicated $\beta = 1$ was near the optimum convergence rate, a result in agreement with stability theory.[11]

The converged flowfield obtained from the incompressible equations with $\beta = 1$ is shown in Fig. 6. These results correspond to the solution after 300 iterations. The residuals have all been reduced below 10^{-6} (see Fig. 5). Figure 6a shows the velocity contours in the nozzle, while Fig. 6b shows the corresponding pressure contours. Both of these results show smooth, wiggle-free contours even though no artificial viscosity was used in their solution. The attainment of smooth velocity contours in incompressible flow is generally relatively easy, but well-behaved pressure contours are a more sensitive test of the capabilities of the algorithm. Another check on the accuracy of the calculation is the degree to which the divergence of the velocity is enforced. In the present calculations, the L_2 norm of the divergence of the velocity has been reduced by five orders of magnitude in 300 steps and, as Fig. 5 suggests, additional iterations would continue to drive it lower.

From these results, we see that if β is chosen as 1, the incompressible equations converge reliably with the Euler-implicit scheme. Comparison with the compressible calculations shows that both systems converge at essentially the same rates if the Mach number in the compressible equations is equal to the pseudo-Mach number of the incompressible system. For the incompressible equations, the pseudo-Mach number is determined by the parameter β and can be chosen arbitrarily. For the compressible equations, the Mach number is determined by the physics of the problem. Here, the Mach number does affect the steady solution, but those Mach numbers leading to incompressible flowfields exhibit very poor convergence. Finally, we note that even in those

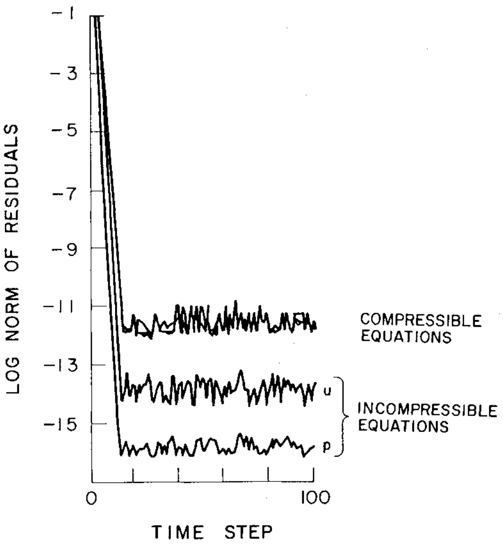

Fig. 1 Convergence of the Euler implicit algorithm for one-dimensional equations (CFL = 10^4).

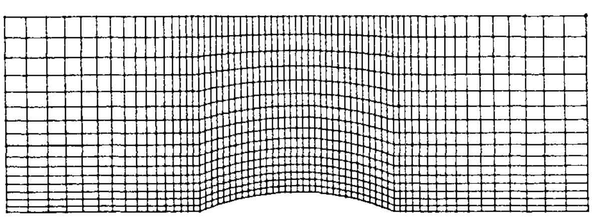

Fig. 2 Nozzle geometry and grid configuration for two-dimensional calculations.

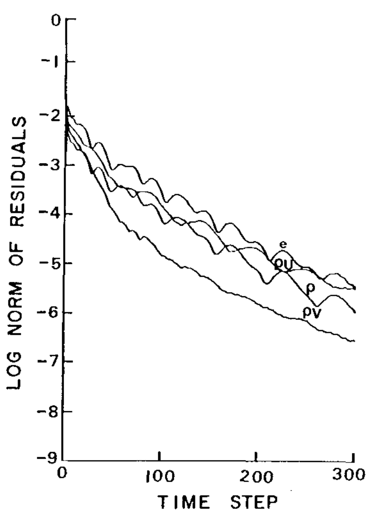

Fig. 3 Convergence of Euler implicit algorithm for two-dimensional compressible equations (inlet Mach number = 0.5, CFL = 5).

regimes where the compressible equations do converge as rapidly as the incompressible ones, the latter are distinctly more economical to use because the vectors in the incompressible equations contain only three components as compared to four for the compressible equations. As a result, the computation time with the incompressible equations is about one-half that with the compressible equations.

Convergence Enhancement at Low Mach Numbers

As noted above, the convergence characteristics of the compressible system are similar to those of the incompressible system at moderate subsonic Mach numbers, but at low Mach numbers the convergence deteriorates rapidly. An important question is, how can reliable convergence be maintained when the Mach number decreases? The answer to this question lies in identifying how the Mach number decrease affects the system. Mach number changes cause at least two phenomena to occur in the equations. First of all, as we noted earlier, the presence of low Mach numbers implies the eigenvalues in both the compressible and incompressible systems become stiff. In compressible flows, the eigenvalues (in a Cartesian system) can be written

$$\lambda(A_c) = [u,u,u(1+1/M_c),u(1-1/M_c)] \tag{9}$$

Clearly, as $M_c \to 0$, the last two eigenvalues approach $\pm\infty$, while the first two remain at unity (if u is nondimensionalized by an appropriate reference velocity).

For the incompressible case, the eigenvalues are

$$\lambda(A_i) = [u,u(1+1/M_i),u(1-1/M_i)] \tag{10}$$

and similar conclusions are reached. Thus, one possibility is that the stiffness of the eigenvalues causes the deterioration of the scheme at low Mach numbers. However, this is refuted by Fig. 1, which shows that for the one-dimensional case stiffness has no effect on convergence for either the compressible or incompressible equations. In addition, numerical experiments in which the time derivatives in the eigenvalues of the compressible equations were scaled by preconditioning by an appropriate matrix to eliminate the stiffness have also verified this. Accordingly, it is necessary to look for another reason for this deterioration.

A second possiblity is that the compressible equations become improperly scaled as the Mach number is reduced to zero and that they must be renormalized in an incompressible sense, as suggested by Briley et al.[26] Although their results show that rescaling does give improvement, their explanation does not appear to address the fundamental difficulty since their argument would imply that low Mach number corrections are needed in the one-dimensional case as well as the two-dimensional case—a conclusion in obvious disagreement with our one-dimensional calculations.

The third possible reason for the deterioration in convergence at low Mach numbers is the effect of the Mach number on the approximate factorization, as first noted by Steger and Kutler.[5] The approximate factorization error in Eq. (5) is best seen by performing the indicated multiplication to obtain

$$\left(I+\Delta t\frac{\partial}{\partial x}A+\Delta t\frac{\partial}{\partial y}B+\Delta t^2\frac{\partial}{\partial x}A\frac{\partial}{\partial y}B\right)\Delta Q = -\Delta t R \tag{11}$$

which shows the actual system being solved. The product AB contains elements whose magnitudes dominate the terms in the original equation and it is this deviation from the physical equations that precludes convergence.

To demonstrate the contamination entering through the approximate factorization, we first transform Eq. (1) to nonconservative form by premultiplying by a matrix T^{-1}, which is defined by the Jacobian,

$$T^{-1} = \frac{\partial \tilde{Q}}{\partial Q} \tag{12}$$

where $\tilde{Q}$ is the vector of unknowns in the nonconservative form,

$$\tilde{Q}_c = (\rho,u,v,p)^T \tag{13}$$

Upon multiplying Eq. (1) by T^{-1}, we obtain

$$\frac{\partial \tilde{Q}_c}{\partial t} + \tilde{A}\frac{\partial \tilde{Q}_c}{\partial x} + \tilde{B}\frac{\partial \tilde{Q}_c}{\partial y} = 0 \tag{14}$$

where $\tilde{A}$ and $\tilde{B}$ are obtained from A and B by the similarity transformations, $\tilde{A} = T^{-1}AT$, and $\tilde{B} = T^{-1}BT$. (In demonstrating the source of the contamination and its removal, we consider only the compressible equations and use the simpler Cartesian coordinates.)

The product term AB introduced into the equations of motion by approximate factorization can be shown to be analogous to the product $\tilde{A}\tilde{B}$. Because the algebra is simpler, we consider this latter product. After expressing $\tilde{A}$ and $\tilde{B}$ in standard form,[9] we obtain

$$\tilde{A}\tilde{B} = \begin{bmatrix} uv & \rho v & \rho u & 0 \\ 0 & uv & c^2 & v/\rho \\ 0 & 0 & uv & u/\rho \\ 0 & \rho c^2 v & \rho c^2 u & uv \end{bmatrix} \tag{15}$$

Now, in the low Mach number limit, the speed of sound $c^2 \gg u^2$. Equation (11) shows that the approximate factorization error adds a large term of the form

$$\Delta t^2 c^2 \frac{\partial^2 v}{\partial x \partial y}$$

to the nonconservative form of the x momentum equation. In fact, this term becomes the dominant term on the left-hand side of the x momentum equation and the time-dependent and convective terms are insignificant. Since the equations are fully coupled, the contamination from this one equation can slow the convergence of the entire system.

A method for minimizing this contamination term is easily

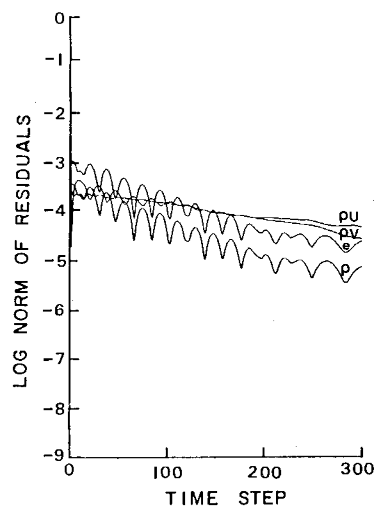

Fig. 4 Convergence of Euler implicit algorithm for two-dimensional compressible equations (inlet Mach number = 0.1, CFL = 2).

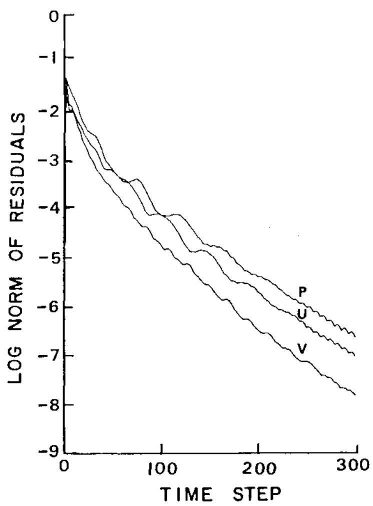

Fig. 5 Convergence of Euler implicit algorithm for incompressible equations ($\beta = 1$, CFL = 5).

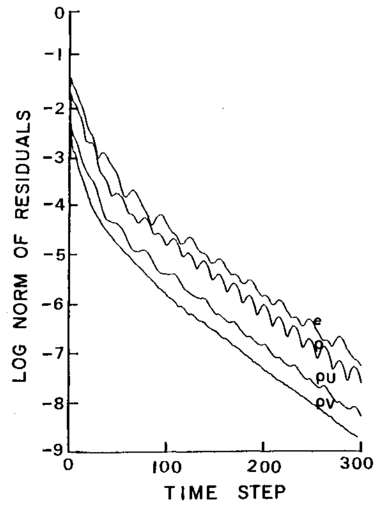

Fig. 7 Convergence of Euler implicit algorithm with preconditioning, compressible equations (inlet Mach number = 0.1, CFL = 5).

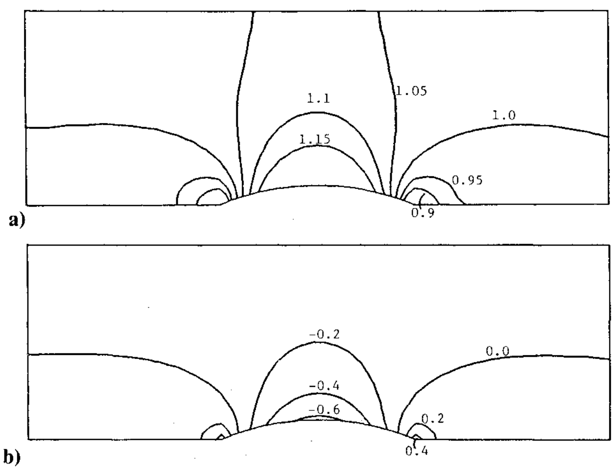

Fig. 6 Velocity and pressure contours from converged solution of incompressible equations: calculation for $\beta = 1$ and CFL = 5; results after 300 iterations: a) velocity contours; b) pressure contours.

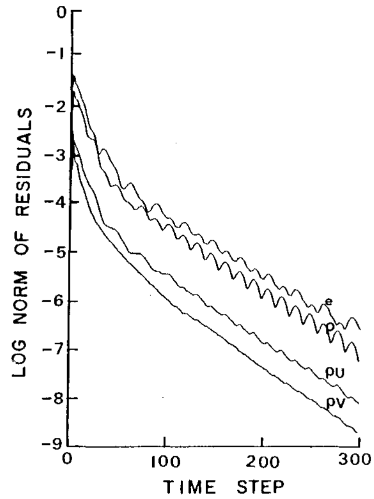

Fig. 8 Convergence of Euler implicit algorithm with preconditioning; compressible equations (inlet Mach number = 0.05, CFL = 5).

obtained. We premultiply the time derivative in Eq. (14) by a preconditioning matrix $\tilde{P}$,

$$\tilde{P} = \mathrm{diag}(1,1,1,M^{-2}) \tag{16}$$

which is defined in such a manner as to remove the large term from the matrix $\tilde{A}\tilde{B}$. With the new time derivative, Eq. (14) becomes

$$\tilde{P}\frac{\partial \tilde{Q}}{\partial t} + \tilde{A}\frac{\partial \tilde{Q}}{\partial x} + \tilde{B}\frac{\partial \tilde{Q}}{\partial y} = 0 \tag{17}$$

The presence of the matrix $\tilde{P}$ changes the approximate factorization error to

$$\tilde{P}^{-1}\tilde{A}\tilde{P}^{-1}\tilde{B} = \begin{bmatrix} uv & \rho v & \rho u & 0 \\ 0 & uv & (u^2+v^2) & vM^2/\rho \\ 0 & 0 & uv & u/\rho \\ 0 & \rho v(u^2+v^2) & \rho u(u^2+v^2)M^2 & uvM^4 \end{bmatrix} \tag{18}$$

None of the elements of this matrix has a magnitude larger than the terms in the original equation.

Before looking at the effect of this preconditioning matrix on the convergence at low Mach numbers, we first transform back to the conservative variables. This transformation is accomplished by premultiplying Eq. (17) by the matrix T defined earlier. This operation gives

$$P\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = 0 \tag{19}$$

where the new matrix P is defined as $P = TPT^{-1}$. For the compressible system, this preconditioning matrix in conservative variables is

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ (u^2+v^2/2)(M^{-2}-1) & u(1-M^{-2}) & v(1-M^{-2}) & M^{-2} \end{bmatrix} \tag{20}$$

Thus, it is seen that this preconditioning has changed only the time derivative of the energy equation.

The effect of using the preconditioning matrix P on the convergence of the compressible equations at low Mach numbers is given on Figs. 7-9. Figure 7 shows the convergence rates for the nozzle flow case with an inlet Mach number of 0.1. As can be seen, this preconditioning allows the solution to converge at a much faster rate than the

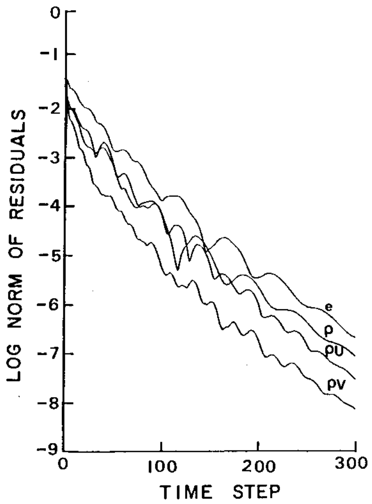

Fig. 9 Convergence of Euler implicit algorithm with preconditioning; compressible equations (inlet Mach number = 0.5, CFL = 5).

original formulation (compare with Fig. 4). Similar results are also noted at $M = 0.05$ as shown in Fig. 8. The converged pressure and velocity fields obtained from this $M = 0.05$ calculation are exactly in agreement with those computed from the incompressible equations previously presented in Fig. 6.

Finally, the effect of preconditioning at the subsonic Mach number of 0.5 was also checked and the results are shown on Fig. 9. Even at this relatively high Mach number, the preconditioning still has a notable favorable effect, as can be seen by comparison with Fig. 3. A further comparison with Fig. 5 also shows that preconditioning to remove the effect of the contamination errors in the approximate factorization allows the compressible equations to converge at almost exactly the same rate as the incompressible equations.

In summary, it is clear that the approximate factorization error is the reason for the slowdown of the convergence of compressible systems as the Mach number is reduced to low values. This error can be eliminated by conditioning the time derivatives in the energy equation, as noted in Eq. (19). It is noted, however, that this modification also removes the stiffness from the equations, so that both adverse effects are removed simultaneously. After preconditioning, the modified conservation system [Eq. (19)] contains the eigenvalues

$$\lambda(A_c) = \left\{u, u, u\left[\frac{1+M^2}{2} \pm \sqrt{\left(\frac{1-M^2}{2}\right)^2 + \frac{u^2+v^2}{u^2}}\right]\right\} \qquad (21)$$

Finally, it is noted that this modification adds only four additional scalar multiplications to each time step calculation and, hence, has no effect on the computation time.

Summary

The application of Euler implicit time-dependent algorithms to the computation of incompressible flowfields has been studied using both the incompressible and low Mach number compressible equations. The application of time-marching concepts to the incompressible equations was accomplished by adding an artificial time derivative to the continuity equation.

Studies of the Euler implicit algorithm for both systems of equations showed that each exhibits similar rates of convergence for a given problem if the Mach number and pseudo-Mach number are equal. This is true for both one- and two-dimensional formulations. The well-known success of implicit schemes in subsonic compressible flows, as well as the difficulties encountered in low Mach number flows, thus suggest that the parameter β should be chosen to give a high subsonic pseudo-Mach number in the incompressible equations. Extensive calculations with the incompressible equations (only a small fraction of which are reported here) have shown that choosing $\beta = 1$ corresponding to a pseudo-Mach number of 0.707 gives fast, reliable convergence similar to that observed with compressible equations. The converged results represent accurate solutions to the incompressible equations. Because the incompressible equations are much simpler than the compressible equations and contain one less equation, they require only about half as much CPU time. The Euler implicit solution of the hyperbolized incompressible equations represents a robust, efficient algorithm for incompressible flows. The extensive knowledge developed over the years with compressible flow calculations can immediately be transferred to the incompressible equations as well.

Although the calculations reported herein are for inviscid flow through a nonstaggered cascade, the algorithm has been tested over a much wider range of problems. Other two-dimensional flows through nozzles and past airfoils with both blunt and sharp leading edges have been computed using both the Euler and the Navier-Stokes equations. In addition, turbulent flowfields through both two- and three-dimensional geometries using a similar formulation have been reported by Chang and Kwak.[7,8]

One of the interesting facts noted in the study of the compressible and incompressible equations was the observation that in one-dimensional problems convergence could be obtained very rapidly (in less than 20 steps) with either system, regardless of the Mach number chosen. Similar calculations in two-dimensional flow show exactly the opposite trend. As the Mach number was lowered below 0.3, the convergence rate slowed dramatically until the algorithm became virtually useless. This difference between one- and two-dimensional analyses arises because of approximate factorization in the presence of stiff eigenvalues. A matrix preconditioning step that eliminates the contamination errors from the approximate factorization was devised and implemented. Calculations with the compressible equations using this preconditioning demonstrated that rapid convergence could be maintained down to a Mach number of 0.05. With preconditioning, the convergence rate becomes independent of Mach number and identical convergence rates are observed for both the compressible and incompressible equations.

The result of the matrix premultiplication is that only the time derivative in the energy equation is altered. The premultiplication is similar to, but distinct from, the one suggested in Ref. 16. An exact comparison between their approach and ours is difficult because Briley et al.[16] assumed constant stagnation enthalpy for their calculations and solved a different set of equations in the incompressible limit rather than for compressible flows. However, we feel that our matrix premultiplication gives a clear, concise explanation of the well-known low Mach number convergence problem. Nevertheless, it would be interesting to attempt to combine the low Mach number scaling concept of Ref. 16 with the approximate factorization modification of the present approach to determine their combined effectiveness.

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